

## 11/22 Surfaces and Calculus

Recall: A surface is  $\vec{S}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$  on domain  $D$ .

Ex. the torus w/ major radius  $a$  and minor radius  $b$

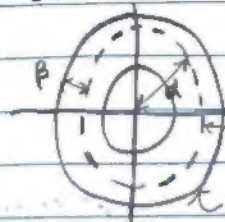
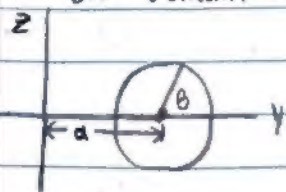
(for  $a > b > 0$ ) is the surface w/ equation

$$\vec{S}(u,v) = \langle (a + b \cos(u)) \cos(v), (a + b \cos(u)) \sin(v), b \sin(u) \rangle$$

on domain  $D = [0, 2\pi] \times [0, 2\pi]$



"mathy donut"



trace at the center of two circles

actual  $z=0$  cross section

$$r = a + b \cos(\theta)$$

### I. Tangent Planes

The tangent plane to surface  $\vec{S}(u,v)$  at input point  $(a,b)$

has normal vector  $\vec{n}(a,b) = \vec{S}_u(a,b) \times \vec{S}_v(a,b)$  where

$$\vec{S}_u = \frac{\partial \vec{S}}{\partial u} = \langle x_u, y_u, z_u \rangle \leftarrow \text{vector of partial derivatives}$$

Ex: Compute the tangent plane to the torus w/ major radius 4 and minor radius 1 at point  $\vec{S}(\frac{3\pi}{4}, \frac{\pi}{4})$

Sol: we want  $\vec{n} \cdot (\vec{x} - \vec{p}) = 0$ , and we're given  $\vec{p} = \vec{S}(\frac{3\pi}{4}, \frac{\pi}{4})$

$$\vec{S}(u,v) = \langle (4 + \cos(u)) \cos(v), (4 + \cos(u)) \sin(v), \sin(u) \rangle \text{ on } [0, 2\pi]^2$$

$$\therefore \vec{p} = \vec{S}(\frac{3\pi}{4}, \frac{\pi}{4}) = \langle (4 + \cos(\frac{3\pi}{4})) \cos(\frac{\pi}{4}), (4 + \cos(\frac{3\pi}{4})) \sin(\frac{\pi}{4}), \sin(\frac{3\pi}{4}) \rangle$$

$$= \langle (4 - 1/\sqrt{2})/\sqrt{2}, (4 - 1/\sqrt{2})/\sqrt{2}, 1/\sqrt{2} \rangle = \langle 4/\sqrt{2} - 1/2, 4/\sqrt{2} - 1/2, 1/\sqrt{2} \rangle$$

Comment: we just need normal at  $(\frac{3\pi}{4}, \frac{\pi}{4})$ , but we'll compute it more generally for use later on

$$\vec{n} = \vec{S}_u \times \vec{S}_v$$

$$\vec{S}_u = \langle -\sin(u) \cos(v), -\sin(u) \sin(v), \cos(u) \rangle \quad \vec{S}_v = \langle -\sin(v)(4 + \cos(u)), \cos(v)(4 + \cos(u)), 0 \rangle$$

$$\vec{n}(u,v) = \vec{S}_u(u,v) \times \vec{S}_v(u,v) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin(u) \cos(v) & -\sin(u) \sin(v) & \cos(u) \\ -\sin(v)(4 + \cos(u)) & \cos(v)(4 + \cos(u)) & 0 \end{vmatrix}$$

$$= (0 - \cos(u) \cos(v)(4 + \cos(u))) \vec{i} - (0 - \cos(u) \sin(v)(4 + \cos(u))) \vec{j} + (-\sin(u) \cos(v) \cos(v)(4 + \cos(u)) - \sin(u) \sin(v) \sin(v)(4 + \cos(u))) \vec{k}$$

$$= -(4 + \cos(u)) \langle \cos(u) \cos(v), \cos(u) \sin(v), \sin(u) \rangle$$

(normal vector to the torus at every input point  $(u,v)$ )



$$\therefore \vec{n}\left(\frac{3\pi}{4}, \frac{\pi}{4}\right) = -(4 + \cos(\frac{3\pi}{4})) \langle \cos(\frac{3\pi}{4})\cos(\frac{\pi}{4}), \cos(\frac{3\pi}{4})\sin(\frac{\pi}{4}), \sin(\frac{3\pi}{4}) \rangle$$

$$= -(4 - 1/\sqrt{2}) \langle -1/\sqrt{2} \cdot 1/\sqrt{2}, -1/\sqrt{2} \cdot 1/\sqrt{2}, 1/\sqrt{2} \rangle$$

$$= -(4 - 1/\sqrt{2}) \langle -1/2, -1/2, 1/\sqrt{2} \rangle$$

$\therefore$  tangent plane to this torus at  $\vec{S}(\frac{3\pi}{4}, \frac{\pi}{4})$  is

$$\vec{n} \cdot (\vec{x} - \vec{p}) = 0 \quad \text{i.e.} \quad \vec{n}\left(\frac{3\pi}{4}, \frac{\pi}{4}\right) \cdot (\vec{x} - \vec{S}(\frac{3\pi}{4}, \frac{\pi}{4})) = 0$$

$$\text{i.e.} \quad -(4 - 1/\sqrt{2}) \langle -1/2, -1/2, 1/\sqrt{2} \rangle \cdot \langle x - 4/\sqrt{2} + 1/2, y - 4/\sqrt{2} + 1/2, z - 1/\sqrt{2} \rangle = 0$$

$$\text{i.e.} \quad -1/2(x - 4/\sqrt{2} + 1/2) - 1/2(y - 4/\sqrt{2} + 1/2) + 1/\sqrt{2}(z - 1/\sqrt{2}) = 0$$

## II Surface Area

The surface area of surface parameterized by  $\vec{S}(u, v)$

$$\text{on domain } D \text{ is } \text{Area}(S) = \iint_D |\vec{S}_u \times \vec{S}_v| \, dA$$

Why that formula? Piecewise-linearly approx. surface  $S$  via parallelograms. Limit sums of those area approximations (See website for geogebra approximations)

Ex: for the torus w/ major radius 4 and minor radius 1,

compute the surface area

$$\text{Sol: } \text{Area}(S) = \iint_D |\vec{S}_u \times \vec{S}_v| \, dA$$

$$\text{from before } \vec{S}_u(u, v) \times \vec{S}_v(u, v) = -(4 + \cos(u)) \langle \cos(u)\cos(v), \cos(u)\sin(v), \sin(u) \rangle$$

So we compute

$$|\vec{S}_u \times \vec{S}_v| = |-(4 + \cos(u))| \sqrt{\cos^2(u)\cos^2(v) + \cos^2(u)\sin^2(v) + \sin^2(u)}$$

$$= |4 + \cos(u)| \sqrt{\cos^2(u)(\cos^2(v) + \sin^2(v)) + \sin^2(u)}$$

$$= 4 + \cos(u)$$

$$\int_{u=0}^{2\pi} \int_{v=0}^{2\pi} 4 + \cos(u) \, dv \, du$$

$$\int_{u=0}^{2\pi} (4 + \cos(u)) [v]_0^{2\pi} \Rightarrow 16\pi^2$$

Exercise: Compute surface area of general torus w/ major radius  $a$  and minor radius  $b$  ( $a > b > 0$ )

### III - Surface Integral

The (surface) integral of function  $f(x, y, z)$  over surface  $S$  parameterized by  $\vec{S}(u, v)$  on domain  $D$  is

$$\iint_S f \, dS = \iint_D f(\vec{S}(u, v)) |\vec{S}_u \times \vec{S}_v| \, dA$$

Note: the correct way to understand " $dS = |\vec{S}_u \times \vec{S}_v| \, dA$ " is via a Jacobian.

$|\vec{S}_u \times \vec{S}_v|$  is the Jacobian of a coordinate change